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DYNAMIC STABILITY OF SPINNING PRE-TWISTED BEAMS

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Abstract—The equations of motion of a spinning pre-twisted beam are formulated using Euler beam theory and the assumed mode method. The equations of motion are then transformed to the standard form of an eigen-value problem for determining the critical spinning speeds corresponding to the divergence behavior of the spinning beam. The effects of pre-twist angle and aspect ratio of the rectangular cross-section of the beam on its stability are investigated. Detailed numerical simulations show that the unstable spinning speed zones for a pre-twisted beam of a given aspect ratio of the cross-section do not become narrower with increased pre-twist angle. The main reason is that for a pre-twisted beam, the critical spinning speeds corresponding to divergence behaviors are no longer the dividing points for separating the speed zones into stable and unstable regions. This type of stability behavior of a pre-twisted beam is different from the stability behavior of a spinning beam without any pre-twisting. Such a spinning beam without pre-twisting is found to have distinct regions of stable and unstable spinning speed zones separated by critical spinning speeds.

1. INTRODUCTION

The dynamic behaviors of spinning beams have been studied extensively in relation to the vibration of rotating shafts, drills, end-mills, boring bars and satellite booms. Likins *et al.* (1973) and Bauer (1980) investigated an Euler beam attached to a rigid base spinning with a constant angular speed. Laurenson (1976) analysed the behavior of a spinning beam having different flexural rigidities in the two principal directions of the cross-section. Leung and Fung (1988) analysed the vibration of spinning Euler beams using the finite element method. Filipich *et al.* (1987) investigated the vibration of a spinning beam with uniform cross-section having only one axis of symmetry. Kane *et al.* (1987) investigated a Timoshenko beam built into a rigid base undergoing general three-dimensional motions. Kammer and Schlack (1987) analysed an Euler beam with a constant spin rate superimposed by small periodic perturbations. The vibrations of pre-twisted beams under axial compressive loads with elastic constraints were investigated by Chen and Liao (1991). The major part of their work was on the determination of natural frequencies for a beam with various combinations of pre-twist angles, spinning speeds, aspect ratio of the cross-section and axial compressive loads. Only the first critical speed zone was presented for a pre-twisted beam with a prescribed aspect ratio of the cross-section. Moreover, they reported that the first critical spinning speed was found to increase for larger pre-twist angle. The first unstable spinning speed zone was also reported to be narrower for a beam with larger pre-twist angle. The other studies related to pre-twisted beams [for example, Carnegie and Thomas (1972); Celep (1985)] were for non-spinning pre-twisted beams.

It is well known that the equations of motion of a spinning Euler beam with circular cross-section, when derived in an inertial coordinate system, are identical to the equations of motion of a non-spinning Euler beam [for example, Han and Zu (1992); Katz *et al.* (1988)]. This finding is important as it implies that the stability behavior of a spinning beam with circular cross-section is independent of the spinning speed. One can conclude further that such a spinning beam will always be stable irrespective of the spinning speed and any prescribed spinning motion. Such a conclusion has also been experimentally and theoretically verified for a spinning Euler beam with the same flexural rigidities in the two principal directions of the cross-section (Lim and Ong, 1992). A spinning beam with equal

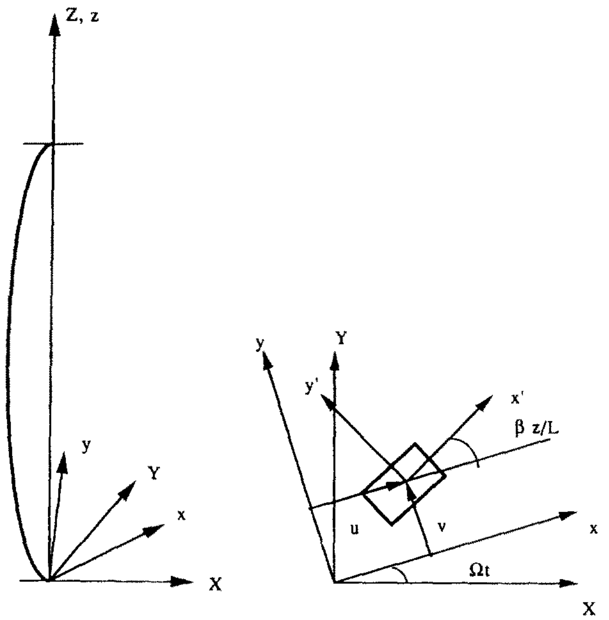

Fig. 1. A spinning pre-twisted beam.

flexural rigidities in the two principal directions of the cross-section was also reported to be always stable by Kammer and Schlack (1987). When the equations of motion of such a spinning beam are derived using a body-fixed coordinate system, the spinning speed of the beam will appear in the equations of motion. Critical speeds in terms of the divergence behavior of the beam can be found. It is important to note that such critical speeds refer only to the divergence behavior, which corresponds to zero real and imaginary parts of the corresponding eigen-values, and do not imply unstable behaviors for the beam, whereas for unstable behaviors, the real part of the corresponding eigen-values will have to be positive. Therefore, the findings obtained from formulations based on body-fixed coordinate systems do not contradict the conclusion drawn from formulations based on inertial coordinate systems. Consequently, an inertial coordinate system is a simpler and better choice for describing the motion of a spinning beam with equal flexural rigidities in the two principal directions of the cross-section. However, for a beam with different flexural rigidities in the two principal directions of the cross-section, a body-fixed coordinate system has to be used to derive the corresponding equations of motion.

In the present study, the equations of motion in matrix form for a spinning pre-twisted beam are formulated using Hamilton's principle and the assumed mode method. The equations of motion are then transformed to the standard form of an eigen-value problem for studying the influence of the pre-twist and aspect ratio of the cross-section on the divergence-type instability of the spinning beam. The stability of a beam with no pre-twisting will be analysed first for comparison with the reported results. The stability of a pre-twisted beam is then analysed in detail based on numerical simulations. It will be shown in the present paper that the stability behavior of a spinning pre-twisted beam is different from the stability behavior of a spinning beam without any pre-twisting. A major finding is that the critical spinning speeds corresponding to divergence behaviors of a pre-twisted beam are no longer the dividing points for separating the speed zones into stable and unstable regions.

2. THEORY AND FORMULATIONS

The present formulation for the kinetic energy and potential energy follows closely the work presented by Chen and Liao (1991). The beam shown in Fig. 1, is a pre-twisted beam of uniform cross-section with overall pre-twist angle β and length L, rotating with a constant angular speed Ω about its longitudinal axis. A body-fixed coordinate system (x, y, z) is moving with the spinning beam in an inertial coordinate system (X, Y, Z). A third coordinate system (x', y', z') is defined to be a local coordinate system along the principal axes of the beam for a particular cross-section. The three Z, z, and z' axes are coincident at all times. The pre-twist angle is assumed to be varying linearly along the beam. The deflections are assumed to be small for the deformation to be governed by Euler beam theory. There is no coupling between torsional and transverse vibrations as the cross-section is rectangular with coincident centroid and shear center.

The deformation of the beam is described by the transverse deflections $u(z, t)$, and $v(z, t)$ in the x and y directions of the beam. The deflections are coupled due to the spinning motion and the fact that the principal axes, which change along the beam, are no longer coincident with the (x, y, z) axes. The kinetic and potential energy are given by (Chen and Liao, 1991)

$$T = \tfrac{1}{2}m\int_0^L \{\dot{u}^2+\dot{v}^2+\Omega^2(u^2+v^2)+2\Omega(u\dot{v}-\dot{u}v)\}\,\mathrm{d}z \tag{1}$$

$$U = \tfrac{1}{2}E\int_0^L \left\{\left(I_{x'}\sin^2\beta\frac{z}{L}+I_{y'}\cos^2\beta\frac{z}{L}\right)\left(\frac{\partial^2 u}{\partial z^2}\right)^2+(I_{x'}-I_{y'})\left(\sin 2\beta\frac{z}{L}\right)\frac{\partial^2 u}{\partial z^2}\frac{\partial^2 v}{\partial z^2}\right.$$
$$\left.+\left(I_{x'}\cos^2\beta\frac{z}{L}+I_{y'}\sin^2\beta\frac{z}{L}\right)\left(\frac{\partial^2 v}{\partial z^2}\right)^2\right\}\mathrm{d}z, \tag{2}$$

where m is the mass of the beam per unit length, E is the Young's modulus and $I_{x'}$ and $I_{y'}$ are the moments of inertia about the x' and y' principal axes in the local coordinate system (x', y', z').

For simplicity in the subsequent derivations and computations, the following dimensionless quantities are introduced:

$$\tau = t\sqrt{\frac{EI_{y'}}{mL^4}} \tag{3}$$

$$\xi = \frac{z}{L} \tag{4}$$

$$\bar{\Omega} = \Omega\sqrt{\frac{mL^4}{EI_{y'}}} \tag{5}$$

$$\kappa = \frac{I_{x'}}{I_{y'}}. \tag{6}$$

Using the assumed mode method, the dimensionless quantities $\bar{u}$ and $\bar{v}$ can be expressed as

$$\bar{u} = \frac{u}{L} = \sum_{i=1}^{n}\bar{p}_i(\tau)\phi_i(\xi) \tag{7}$$

$$\bar{v} = \frac{v}{L} = \sum_{i=1}^{n}\bar{q}_i(\tau)\phi_i(\xi), \tag{8}$$

where ϕ_ξ are spatial functions that satisfy the boundary conditions at the two ends of the beam. For a beam simply supported at both ends, the assumed functions are

$$\phi_i(\xi) = \sqrt{2}\sin i\pi\xi. \tag{9}$$

The assumed forms of $\bar{u}$ and $\bar{v}$ enable the kinetic energy and the potential energy to be expressed in matrix form as follows:

$$T = \tfrac{1}{2}\mathring{\bar{\mathbf{p}}}^T\bar{\mathbf{H}}\mathring{\bar{\mathbf{p}}} + \tfrac{1}{2}\mathring{\bar{\mathbf{q}}}^T\bar{\mathbf{H}}\mathring{\bar{\mathbf{q}}} + \bar{\Omega}\bar{\mathbf{p}}^T\bar{\mathbf{H}}\mathring{\bar{\mathbf{q}}} - \bar{\Omega}\mathring{\bar{\mathbf{p}}}^T\bar{\mathbf{H}}\bar{\mathbf{q}} + \tfrac{1}{2}\bar{\Omega}^2\bar{\mathbf{p}}^T\bar{\mathbf{H}}\bar{\mathbf{p}} + \tfrac{1}{2}\bar{\Omega}^2\bar{\mathbf{q}}^T\bar{\mathbf{H}}\bar{\mathbf{q}} \tag{10}$$

$$U = \tfrac{1}{2}\bar{\mathbf{p}}^T\bar{\mathbf{K}}_1\bar{\mathbf{p}} + \tfrac{1}{2}\bar{\mathbf{q}}^T\bar{\mathbf{K}}_2\bar{\mathbf{q}} - \tfrac{1}{2}\bar{\mathbf{p}}^T\bar{\mathbf{K}}_3\bar{\mathbf{q}}, \tag{11}$$

where

$$(\bar{\mathbf{H}})_{ij} = \int_0^1 \phi_i\phi_j \,\mathrm{d}\xi \tag{12}$$

$$(\bar{\mathbf{K}}_1)_{ij} = \int_0^1 (\kappa\sin^2\beta\xi + \cos^2\beta\xi)\phi_i''\phi_j'' \,\mathrm{d}\xi \tag{13}$$

$$(\bar{\mathbf{K}}_2)_{ij} = \int_0^1 (\kappa\cos^2\beta\xi + \sin^2\beta\xi)\phi_i''\phi_j'' \,\mathrm{d}\xi \tag{14}$$

$$(\bar{\mathbf{K}}_3)_{ij} = \int_0^1 (1-\kappa)(\sin 2\beta\xi)\phi_i''\phi_j'' \,\mathrm{d}\xi. \tag{15}$$

The matrix $\bar{\mathbf{H}}$ is equal to the identity matrix due to the orthogonality of the assumed beam functions. The vectors $\bar{\mathbf{p}}$, $\mathring{\bar{\mathbf{p}}}$ and $\mathring{\bar{\mathbf{q}}}$ are $n\times 1$ column vectors consisting of $\bar{p}_i$, $\mathring{\bar{p}}_i$, $\bar{q}_i$ and $\mathring{\bar{q}}_i$, respectively. The notation ($''$) denotes second partial derivative with respect to ξ.

The resulting dimensionless equations of motion are

$$\bar{\mathbf{H}}\mathring{\mathring{\bar{\mathbf{p}}}} + \bar{\mathbf{K}}_1\bar{\mathbf{p}} - \tfrac{1}{2}\bar{\mathbf{K}}_3\bar{\mathbf{q}} - 2\bar{\Omega}\bar{\mathbf{H}}\mathring{\bar{\mathbf{q}}} - \bar{\Omega}^2\bar{\mathbf{H}}\bar{\mathbf{p}} = 0 \tag{16}$$

$$\bar{\mathbf{H}}\mathring{\mathring{\bar{\mathbf{q}}}} + \bar{\mathbf{K}}_2\bar{\mathbf{q}} - \tfrac{1}{2}\bar{\mathbf{K}}_3\bar{\mathbf{p}} + 2\bar{\Omega}\bar{\mathbf{H}}\mathring{\bar{\mathbf{p}}} - \bar{\Omega}^2\bar{\mathbf{H}}\bar{\mathbf{q}} = 0. \tag{17}$$

The two matrix equations are coupled due to the spinning motion and the pre-twisting of the beam. The equations of motion can be presented in the matrix form

$$\mathbf{I}\ddot{\mathbf{a}} + \mathbf{B}\dot{\mathbf{a}} + \mathbf{C}\mathbf{a} = \mathbf{0}, \tag{18}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{0}$ is the zero matrix and

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & -2\bar{\Omega}\mathbf{I} \\ 2\bar{\Omega}\mathbf{I} & \mathbf{0} \end{bmatrix} \tag{19}$$

$$\mathbf{C} = \begin{bmatrix} \bar{\mathbf{K}}_1 - \bar{\Omega}^2\mathbf{I} & -\tfrac{1}{2}\bar{\mathbf{K}}_3 \\ -\tfrac{1}{2}\bar{\mathbf{K}}_3 & \bar{\mathbf{K}}_2 - \bar{\Omega}^2\mathbf{I} \end{bmatrix}. \tag{20}$$

Equation (18) can be cast in the form

$$\dot{\mathbf{b}} = \mathbf{G}\mathbf{b} \tag{21}$$

with

$$\mathbf{b} = \begin{bmatrix} \dot{\mathbf{a}} \\ \mathbf{a} \end{bmatrix} \tag{22}$$

$$\mathbf{G} = \begin{bmatrix} -\mathbf{B} & -\mathbf{C} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{23}$$

For a non-trivial solution, one has the characteristic equation

$$\det|\mathbf{G}-\zeta\mathbf{I}| = 0. \tag{24}$$

The quantities ζ are the complex eigen-values which can be expressed as

$$\zeta = \rho \pm i\omega, \tag{25}$$

where i is the imaginary unit, ρ is the dimensionless damping and ω is the oscillatory dimensionless eigen-frequencies. The condition of $\rho = \omega = 0$ gives the divergence limit. If ρ is positive, the motion will be unstable.

For $\zeta = 0$, the divergence equation given by eqn (24) can be reduced to

$$\det|\mathbf{C}| = 0. \tag{26}$$

For a given pre-twist angle β and cross-section κ, the square of the critical dimensionless spinning speeds of the beam can be found from the eigen-values of the following eigen-value problem from eqn (26):

$$\det|\mathbf{A}-\bar{\Omega}^2\mathbf{I}| = 0, \tag{27}$$

where

$$\mathbf{A} = \begin{bmatrix} \bar{\mathbf{K}}_1 & -\frac{1}{2}\bar{\mathbf{K}}_3 \\ -\frac{1}{2}\bar{\mathbf{K}}_3 & \bar{\mathbf{K}}_2 \end{bmatrix}. \tag{28}$$

The square of the critical speeds are just the eigen-values of the matrix $\mathbf{A}$. The eigen-values can be computed easily using any commercial computer software for matrix computations.

3. RESULTS AND SIMULATIONS

The present formulation enables the critical speeds of a pre-twisted spinning beam to be determined conveniently from eqn (27) using any commercial software package (for example, PCMATLAB) for matrix computations. The stability of the spinning beam can also be determined easily by examining the sign of the real parts of the eigen-values of the matrix $\mathbf{G}$ in eqn (24). The case of a spinning beam with no pre-twisting is first examined for a beam simply supported at both ends. Numerical results for other boundary conditions can be easily generated using the appropriate assumed functions.

For a beam with no pre-twisting, the pre-twist angle β is equal to zero. Consequently, the matrix $\bar{\mathbf{K}}_3$ is equal to a zero matrix. The eigen-values of the matrix $\mathbf{A}$ are just the eigen-values of matrix $\bar{\mathbf{K}}_1$ and $\bar{\mathbf{K}}_2$. Moreover, the matrix $\bar{\mathbf{K}}_1$ is independent of κ from eqn (13). The eigen-values for $\bar{\mathbf{K}}_1$ are therefore equal to π^4, $(2\pi)^4 \ldots (n\pi)^4$ for simply supported beam. The matrix $\bar{\mathbf{K}}_2$ is also equal to $\bar{\mathbf{K}}_1$ when κ is equal to one. The dimensionless critical speeds $\bar{\Omega}$ for a simply supported spinning beam with equal flexural rigidities in the two principal directions ($\kappa = 1$) are therefore equal to π^2, $(2\pi)^2 \ldots (n\pi)^2$. When κ decreases, the eigen-values of $\bar{\mathbf{K}}_2$ are found to be on the same decreasing trend. These changes in the critical

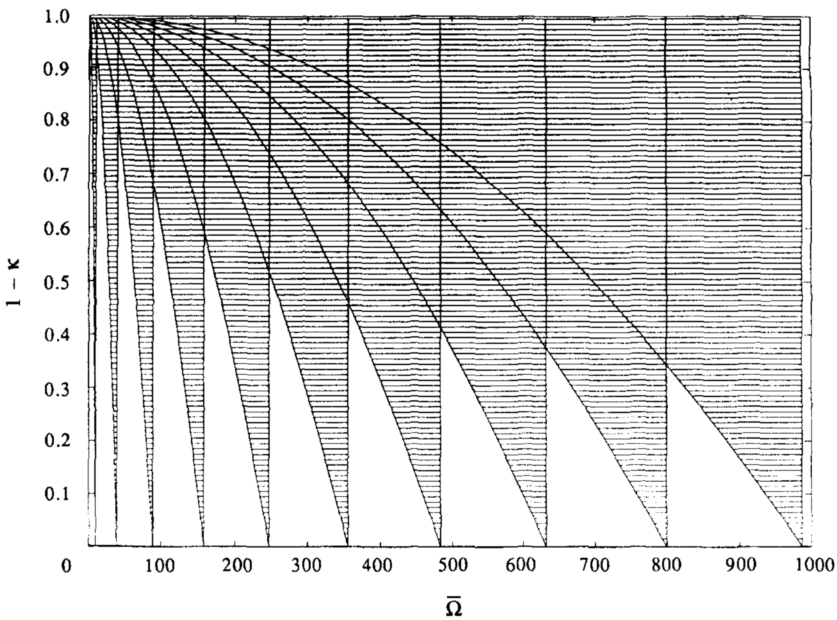

Fig. 2. Critical non-dimensional spinning speeds for a non pre-twisted beam.

speeds due to $\bar{\mathbf{K}}_2$ are indicated by the curved solid lines in Fig. 2. As $\bar{\mathbf{K}}_1$ is independent of κ, the critical speeds corresponding to $\bar{\mathbf{K}}_1$ remain unchanged and are indicated by vertical straight lines shown in Fig. 2. The shaded regions, indicated in Fig. 2, are found to be corresponding to unstable behaviors with positive real part for the corresponding eigen-values. Only the first 10 unstable regions are indicated in Fig. 2. It can be seen that for a spinning beam with a prescribed shape for the cross-section, the motion is unstable when the spinning speed of the beam lies within certain intervals which are bounded by the critical speeds. This finding is in complete agreement with the reported finding by Ariaratnam (1965). It can also be seen from Fig. 2 that the stable regions become smaller and smaller with increased divergence of the flexural rigidities for the two principal directions of the cross-section. Moreover, for a spinning beam with a large difference in the flexural rigidities in the two principal directions of the cross-section (i.e. the upper portion of Fig. 2), there exists a critical speed above which the motion is always unstable.

For a pre-twisted spinning beam with non-zero β, the matrix $\bar{\mathbf{K}}_3$ is no longer a zero matrix. The critical speeds corresponding to divergence behaviors, indicated by solid curved lines, are shown in Figs 3–7 for $\beta = 10°$, $45°$, $90°$, $180°$ and $360°$. The shaded regions, bounded by the two curves originating from the same point on the horizontal axis with $\kappa = 1$, are found to correspond to unstable behaviors with positive real parts for the corresponding eigen-values. These regions are found to be smaller for a spinning beam with larger pre-twist angles. Moreover, for $\kappa = 1$, the critical speeds of the spinning beam are found to be independent of β, which can be easily verified from the definitions of $\bar{\mathbf{K}}_1$ and $\bar{\mathbf{K}}_2$. It is very tempting to conclude that the non-shaded regions will be stable as the lines corresponding to divergence behaviors in Fig. 2 for a spinning beam without pre-twisting are found to be the dividing lines for stable and unstable regions. However, a detailed checking of the sign of the eigen-values of these non-shaded regions show otherwise. For example, for the case of a beam with pre-twist angle of 90° shown in Fig. 5, some of the eigen-values are found to possess large positive real parts for κ smaller than 0.6, excluding some distinct values of κ that correspond to divergence behaviors, and also for a tiny region of κ between 0.89 and 0.91 when the dimensionless spinning speed $\bar{\Omega}$ is 250. The upper portions of the non-shaded regions, corresponding to a spinning beam with large differences in the flexural rigidities in the two principal directions of the cross-section, are found to be more likely the unstable regions. The lower portions, on the other hand, are found to be more likely the stable regions. For these pre-twisted spinning beams, the critical spinning speeds corresponding to divergence behaviors are no longer the dividing points for sepa-

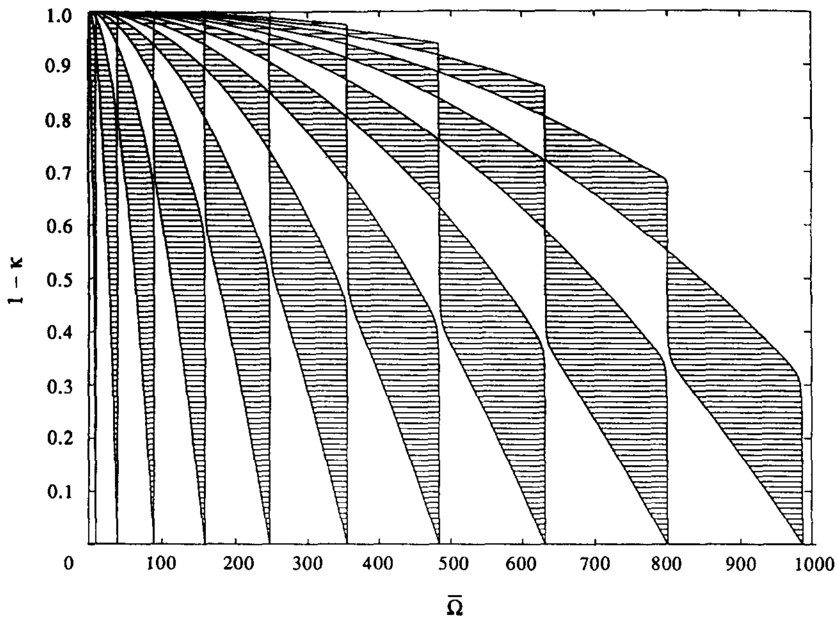

Fig. 3. Critical non-dimensional spinning speeds for a pre-twisted beam with $\beta = 10°$.

rating the speed zones into stable and unstable regions. This is not entirely surprising as there is no reason as to why the sign of the real parts of the eigen-values should be different on the two sides of the line corresponding to divergence behaviors with zero eigen-values. Moreover, if one examines Figs 3–7 closely, one can find that the narrowest gaps between the shaded regions decrease with smaller pre-twist angle for the spinning beam. When the pre-twist angle is close to zero, the solid lines corresponding to divergence behaviors will approach the corresponding solid lines for the case of a non pre-twisted beam shown in Fig. 2. For example, the solid lines for the divergence behaviors shown in Fig. 3 for $\beta = 10°$ are very close to those lines shown in Fig. 2 especially for small and moderate spinning speeds. For the upper portions of the non-shaded regions, it is not likely for a non-pre-twisted spinning beam to change from unstable to stable behavior with a small amount of pre-twisting.

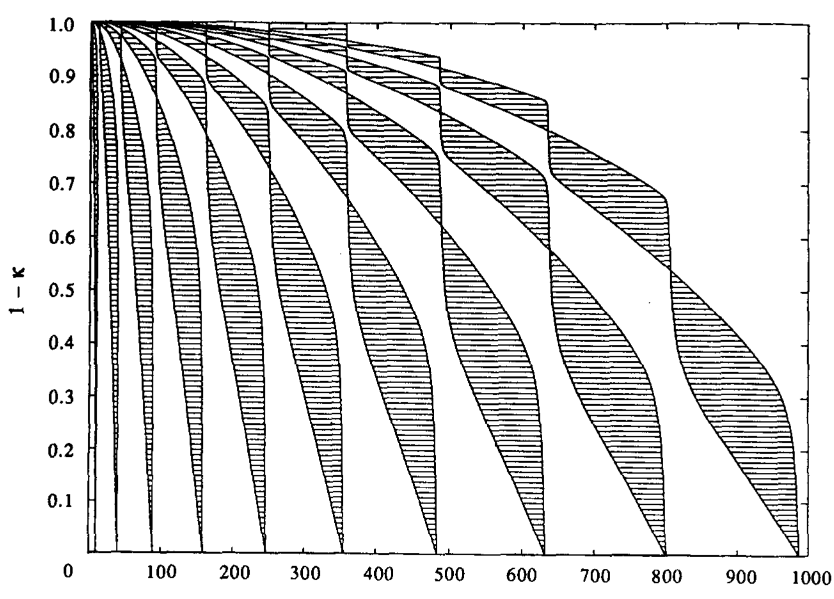

Fig. 4. Critical non-dimensional spinning speeds for a pre-twisted beam with $\beta = 45°$.

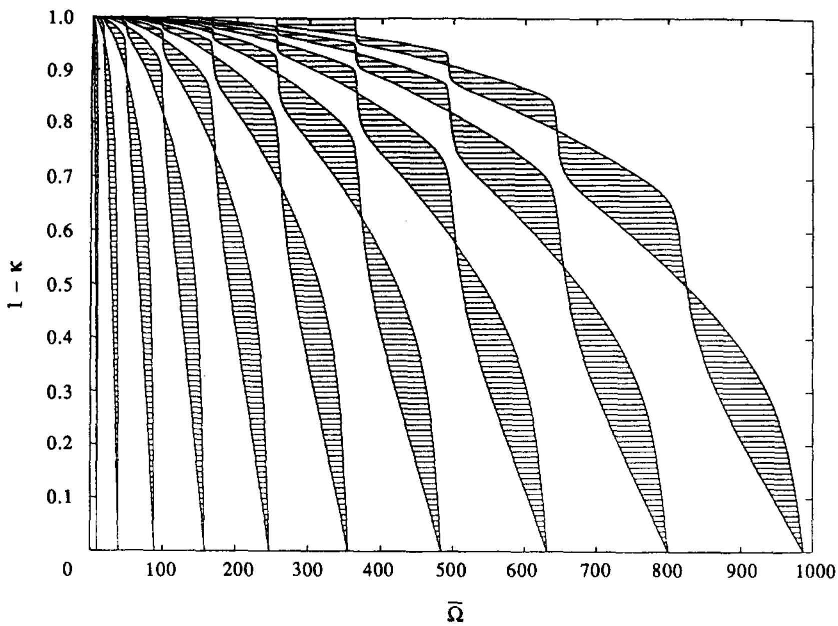

Fig. 5. Critial non-dimensional spinning speeds for a pre-twisted beam with $\beta = 90^{\circ}$.

4. CONCLUSION

The equations of motion in matrix form are formulated for the dynamic behavior of a spinning pre-twisted beam based on Hamilton's principle and the assumed mode method. The equations of motion are then transformed to the standard form of an eigenvalue problem for determining the critical dimensionless spinning speeds corresponding to the divergence-type instability of the beam. Results of numerical simulations are presented for various combinations of pre-twist angle and prescribed cross-section of the beam. An advantage of the present formulation is that the critical speeds and the stability for a spinning beam can be easily determined using any commercial package for matrix computation. The numerical results for non pre-twisted spinning beam are found to be in agreement with the reported works. For pre-twisted spinning beams, the stability behavior is found to be drastically different from that of a spinning beam without any pre-twisting. A

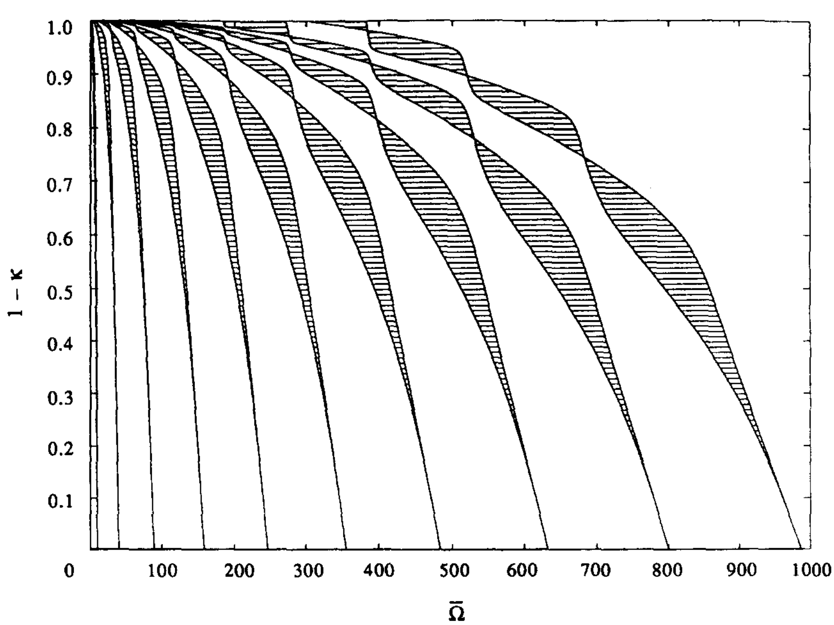

Fig. 6. Critical non-dimensional spinning speeds for a pre-twisted beam with $\beta = 180^{\circ}$.

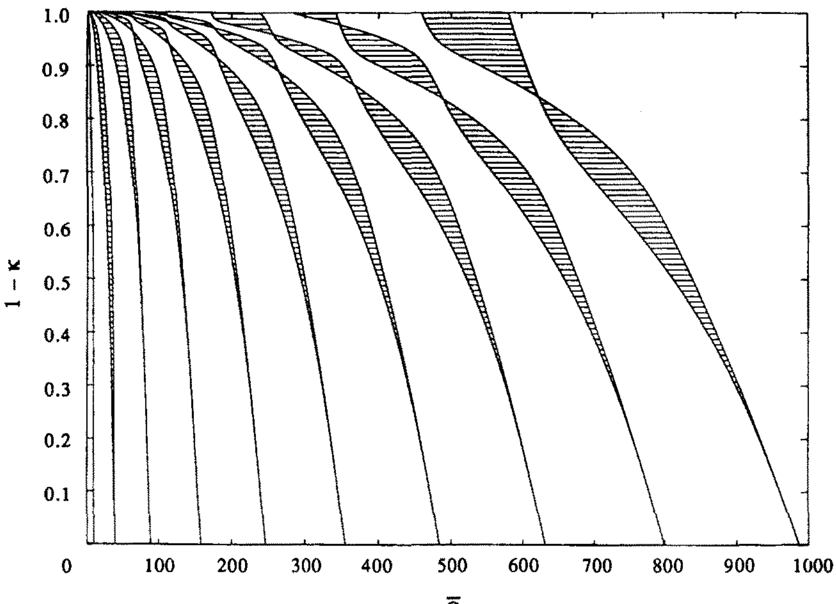

Fig. 7. Critical non-dimensional spinning speeds for a pre-twisted beam with $\beta = 360°$.

major finding is that the critical spinning speeds corresponding to divergence behaviors of a pre-twisted beam are no longer the dividing points for separating the speed zones into stable and unstable regions.

REFERENCES

Ariaratnam, S. T. (1965). The vibration of unsymmetrical rotating shaft. *J. Appl. Mech.* **32,** 157–162.

Bauer, H. F. (1980). Vibration of a rotating beam. Part 1: Orientation in the axis of rotation. *J. Sound Vibr.* **72,** 177–178.

Carnegie, W. and Thomas, J. (1971). The coupled bending–bending vibration of pre-twisted tapered bladings. *J. Engng Industry* **75,** 255–266.

Celep, Z. (1985). Dynamic stability of pre-twisted columns under periodic axial loads. *J. Sound Vibr.* **103,** 35–42.

Chen, M. L. and Liao, Y. S. (1991). Vibration of pre-twisted spinning beams under axial compressive loads with elastic constraints. *J. Sound Vibr.* **147,** 497–513.

Filipich, C. P., Naurizi, M. J. and Rosales, M. B. (1987). Free vibrations of a spinning uniform beam with ends elastically restrained against rotation. *J. Sound Vibr.* **116,** 475–482.

Han, R. P. S. and Zu, J. W. Z. (1992). Modal analysis of rotating shafts: a body-fixed axis formulation approach. *J. Sound Vibr.* **156,** 1–16.

Kammer, D. C. and Schlack, A. L. (1987). Effects of non-constant spin rate on the vibration of a rotating beam. *J. Appl. Mech.* **54,** 305–310.

Kane, T. R., Ryan, R. R. and Banerjee, A. K. (1987). Dynamics of a cantilever beam attached to a moving base. *J. Dynamics Guidance Control* **10,** 139–151.

Katz, R., Lee, C. W., Ulsoy, A. G. and Scott, R. A. (1988). The dynamic response of a rotating shaft subject to a moving load. *J. Sound Vibr.* **122,** 131–148.

Laurenson, R. M. (1976). Modal analysis of rotating flexible structure. *AIAA J.* **14,** 1444–1450.

Leung, A. Y. T. and Fung, T. C. (1988). Spinning finite elements. *J. Sound Vibr.* **125,** 523–537.

Likins, P. W., Barbera, F. J. and Baddeley, V. (1973). Mathematical modeling of spinning elastic bodies for modal analysis. *AIAA J.* **11,** 1251–1258.

Lim, S. P. and Ong, E. H. (1992). Dynamics of a flexible circular beam. *Proc. Int. Conf. Comput. Engng Sci.* December 17–22, Hong Kong.